

Time: 3 Hours

Total Marks: 100

Section-A

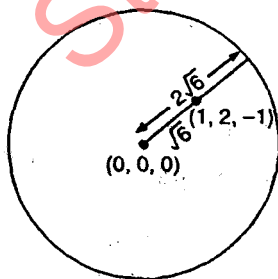
All parts of this question are compulsory.

$$(2 \times 10) = 20$$

Q.1. (a) If $f(x) = f(0) + k f_1(0) + \frac{k^2}{L^2} f_2(\theta k)$,
 $0 < \theta < 1$ then the value of θ when $k = 1$ and $f(x) = (1-x)^{5/2}$ is given as

Ans.

Q.1. (b) The short distance from the point $(1, 2, -1)$ to the sphere $x^2 + y^2 + z^2 = 24$ shall be

Ans. $\sqrt{6}$ Justification : Given point = $(1, 2, -1)$ Given sphere is $x^2 + y^2 + z^2 = 24$ Centre of given sphere is $(0, 0, 0)$ Its radius = $\sqrt{24} = 2\sqrt{6}$ Distance of $(0, 0, 0)$ to $(1, 2, -1)$ is

$$\begin{aligned} d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \\ &= \sqrt{(1-0)^2 + (2-0)^2 + (-1-0)^2} \\ &= \sqrt{1+4+1} = \sqrt{6} \end{aligned}$$

 \therefore distance from centre is less than radius. \therefore Point lies inside the sphere.

\therefore Distance from surface = Radius - Distance from centre
 $= 2\sqrt{6} - \sqrt{6} = \sqrt{6}$

Q.1. (c) The jacobian $J\left(\frac{u, v}{x, y}\right)$ for $u = e^x \sin y$,
 $v = x \log \sin y$ shall be

Ans. $e^x \cos y (x - \log \sin y)$

$$\text{Justification : } J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} e^x \sin y & e^x \cos y \\ \log \sin y & x \frac{\cos y}{\sin y} \end{vmatrix}$$

$$\begin{aligned} \Rightarrow J\left(\frac{u, v}{x, y}\right) &= xe^x \cos y - e^x \cos y \log \sin y \\ &= e^x \cos y (x - \log \sin y) \end{aligned}$$

Q.1. (d) For the curve $ay^2 = x^2(a-x)$, which of the following statement(s) is/are Incorrect?

- (i) Curve passes through origin
- (ii) Curve is symmetrical about y-axis
- (iii) Curve has two branches
- (iv) Curve has no tangents at origin

Ans. (i) Correct

(ii) In correct

(iii) Correct

(iv) In Correct

Q.1. (e) If P and Q are non-singular matrices then for Matrix M , which of the following statements are correct?

(i) $\text{Rank}(PMQ) > \text{Rank } M$

(ii) $\text{Rank}(PMQ) = \text{Rank } M$

(iii) $\text{Rank}(PMQ) < \text{Rank } M$

(iv) $\text{Rank}(PMQ) > \text{Rank } M + \text{Rank}(PQ)$

Ans. (i) In Correct

(ii) Correct

(iii) In Correct

(iv) In Correct

Q.1. (f) If λ is an eigen value of the matrix ' M ' then for the matrix $(M - \lambda I)$, which of the following statement(s) is/are correct?

(i) Skew symmetric

(ii) Non singular

(iii) Singular

(iv) None of these

Indicate True/False for the following statements

Ans. (i) In Correct

(ii) In Correct

(iii) Correct

(iv) In Correct

Q. 1. (g) For $\int_0^\infty \int_x^\infty f(x, y) dx dy$, the change of order of integration is

(i) $\int_0^\infty \int_x^\infty f(x, y) dx dy$, True/False

(ii) $\int_x^\infty \int_0^\infty f(x, y) dx dy$, True/False

(iii) $\int_0^\infty \int_0^y f(x, y) dx dy$, True/False

(iv) $\int_0^\infty \int_0^x f(x, y) dx dy$, True/False

Ans. (i) In Correct

(ii) In Correct

(iii) Correct

(iv) In Correct

Justification: Given integral $I = \int_0^\infty \int_x^\infty f(x, y) dx dy$

Region of integration is shown in Figure:

Changing the order of integration. We take strip parallel to x axis.

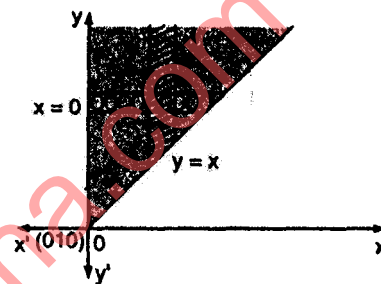


Fig.

Now, strip are

x varies from 0 to y

y varies from 0 to ∞

$$\therefore \int_0^\infty \int_x^\infty f(x, y) dy dx = \int_{y=0}^\infty \int_{x=0}^y f(x, y) dx dy$$

Q.1. (h) The value of $\sqrt{-\frac{1}{2}}$ is given by

(i) $\sqrt{\pi}$, True/False

(ii) $2\sqrt{\pi}$, True/False

(iii) $-\sqrt{\pi}$, True/False

(iv) $-2\sqrt{\pi}$, True/False

Pick up the correct option from the following:

Ans. (i) In Correct

(ii) In Correct

(iii) In Correct

(iv) Correct

Justification : $\sqrt{n+1} = n\sqrt{n}$

$$\Rightarrow \sqrt{n} = \frac{\sqrt{n+1}}{n}$$

$$\Rightarrow \frac{\sqrt{-1}}{2} = \frac{\sqrt{\frac{-1}{2}+1}}{-\frac{1}{2}} = \left[\text{Putting } n = -\frac{1}{2} \right]$$

$$\Rightarrow \sqrt{-\frac{1}{2}} = -2\sqrt{\frac{1}{2}}$$

$$= -2\sqrt{\pi} \quad \left[\frac{1}{2} = \sqrt{\pi} \right]$$

Q.1. (i) If \vec{F} is the velocity of a fluid particle then

$\int_C \vec{F} \cdot d\vec{r}$ represents

- (i) Work done
- (ii) Circulation
- (iii) Flux
- (iv) Conservative field

Ans. Work done

Q. 1. (j) The value of $\iint_S \vec{F} \cdot \vec{n} \, dS$, where

$\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$; a, b, c being constants is given by

$$(i) \frac{\pi}{3}(a+b+c)$$

$$(ii) \frac{4\pi}{3}(a+b+c)$$

$$(iii) 2\pi(a+b+c)$$

$$(iv) \pi(a+b+c)$$

Ans. The given question is wrong as surface is not given.

Section-B

Attempt any three parts of the following :

$$(10 \times 3) = 30$$

Q.2. (a) Determine the values of 'a' and 'b' for which the following system of equations has

$$3x + 5y - az = 7,$$

$$x - by + 4z = -3,$$

$$ax + 4y - 5z = 4$$

(i) No solution

(ii) A unique solution

(iii) Infinite no. of solutions

Ans. Given system is

$$3x + 5y - az = 7$$

$$x + by + 4z = -3$$

$$ax + 4y - 5z = 4$$

Augmented matrix $[A : B]$

$$= \begin{bmatrix} 3 & 5 & -a & : & 7 \\ 1 & -b & 4 & : & -3 \\ a & 4 & -5 & : & 4 \end{bmatrix}$$

Which is not reducible to Conclude form

\therefore Given question is wrong.

Q.2. (b) Find the value of $D^n \{x^{n-1} \log x\}$,

$$D^n = \frac{d^n}{dx^n}$$

Ans. We want to find $D^n \{x^{n-1} \log x\}$

let, $y = x^{n-1} \log x$

Differentiating (1) w.r.t x, we get,

$$y_1 = (n-1)x^{n-2} \log x + x^{n-1} \frac{1}{x}$$

$$\Rightarrow xy_1 = (n-1)x^{n-1} \log x + x^{n-1}$$

$$\Rightarrow xy_1 = (n-1)y + x^{n-1} \quad \dots(2)$$

$$[x^{n-1} \log x = y]$$

Differentiating (2) $n-1$ times by labinitz's theorem,

$$= [xy_n + {}^{n-1}C_1 (1) y_{n-1}]$$

$$= (n-1) y_{n-1} + \frac{d^{n-1}}{dx^{n-1}} (x^{n-1})$$

$$\Rightarrow xy_n + (n-1)y_{n-1} = (n-1)y_{n-1} + \frac{d^{n-1}}{dx^{n-1}} (x^{n-1})$$

$$\left[\begin{array}{l} \because r_{q_1} = r \\ {}^{n-1}C_1 = n-1 \end{array} \right]$$

$$\Rightarrow xy_n = \frac{d^{n-1}}{dx^{n-1}} (x^{n-1})$$

$$\Rightarrow xy_n = (n-1)!$$

$$\left[\because \frac{d^n}{dx^n} (x^n) = n! \right]$$

$$\Rightarrow y_n = \frac{(n-1)!}{x}$$

$$D^n (x^{n-1} \log x) = \frac{(n-1)!}{x}$$

Ans.

Q.2. (c) If $u = \frac{(x+y)}{z}$, $v = \frac{(y+z)}{x}$,

$$w = \frac{y(x+y+z)}{(xz)}$$

then show that u, v, w are not independent and find the relation between them.

Ans. Here, $u = \frac{x+y}{z}$,

$$v = \frac{y+z}{x},$$

$$w = \frac{y(x+y+z)}{xz}$$

B show that, z, y, x and w are not independent, we

shall show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$

Now,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left(\frac{x+y}{z} \right) = \frac{1}{z}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\frac{x+y}{z} \right) = \frac{1}{z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z} \left(\frac{x+y}{z} \right) = -(x+y) \frac{1}{z^2}$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x} \left(\frac{y+z}{x} \right) = -\frac{y+z}{x^2}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y} \left(\frac{y+z}{x} \right) = \frac{1}{x}$$

$$\frac{\partial v}{\partial z} = \frac{\partial}{\partial z} \left(\frac{y+z}{x} \right) = \frac{1}{x}$$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x} \left(\frac{y(x+y+z)}{xz} \right) = \frac{y}{z}$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y} \left(\frac{y(x+y+z)}{xz} \right) = \frac{x+y+z}{xz}$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z} \left(\frac{y(x+y+z)}{xz} \right) = -\frac{y(x+y+z)}{xz^2}$$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x} \left[\frac{y(x+y+z)}{x^2} \right] = \frac{y}{z} \frac{\partial}{\partial x} \left[\frac{x+y+z}{x} \right]$$

$$= \frac{y}{z} \frac{\partial}{\partial x} \left[1 + \frac{y+z}{x} \right]$$

$$= \frac{y}{z} \left[-\left(\frac{y+z}{x^2} \right) \right]$$

$$= -\frac{y(y+z)}{x^2 z}$$

$$= \frac{\partial w}{\partial y} = \frac{\partial}{\partial y} \left[\frac{y(x+y+z)}{xz} \right]$$

$$= \frac{1}{xz} \frac{\partial}{\partial y} [y(x+y+z)]$$

$$\Rightarrow \frac{\partial w}{\partial y} = \frac{1}{xz} [x+2y+z] = \frac{x+2y+z}{xz}$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z} \left[\frac{y(x+y+z)}{xz} \right]$$

$$= \frac{y}{x} \frac{\partial}{\partial z} \left[\frac{x+y}{z} + 1 \right]$$

$$= -\frac{y(x+y)}{xz^2}$$

$$\therefore \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{z} & \frac{1}{z} & -\frac{(x+y)}{z^2} \\ -\frac{(y+z)}{x^2} & \frac{1}{x} & \frac{1}{x} \\ -\frac{y(y+z)}{x^2 z} & \frac{x+2y+z}{xz} & -\frac{y(x+y)}{xz^2} \end{vmatrix}$$

Taking $\frac{1}{z^2}$, $\frac{1}{x^2}$ and $\frac{1}{x^2 z^2}$ common from R_1, R_2 and R_3 respectively, we get,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{1}{z^2} \frac{1}{x^2} \frac{1}{x^2 z^2} \begin{vmatrix} 1 & 1 & -(x+y) \\ -(y+z) & 1 & 1 \\ -y(y+z) & x+2y+z & -y(x+y) \end{vmatrix}$$

$$\begin{vmatrix} z & z & -(x+y) \\ -(y+z) & x & x \\ -yz(y+z) & xz(x+2y+z) & -xy(x+y) \end{vmatrix} = \frac{1}{x^4 z^4}$$

$$\begin{vmatrix} x+y+z & x+y+z & -(x+y) \\ -(x+y+z) & 0 & x \\ xy(x+y) - yz(y+z) & xz(x+2y+z) + xy(x+y) & -xy(x+y) \end{vmatrix}$$

Now, $xy(x+y) - yz(y+z) = y[x(x+y) - z(y+z)]$
 $= y(x^2 + xy - yz - z^2)$
 $= y[x^2 - z^2 + xy + xy - yz]$
 $= y[(x+z)(x-z) + y(x-z)]$
 $= y[(x-z)(x+y+z)]$
 $= y(x-z)(x+y+z)$

Also, $xz(x+2y+z) + xy(x+y)$
 $= x[zx + 2yz + z^2 + xy + y^2]$
 $= x[y^2 + z^2 + 2yz + zx + xy]$
 $= x[(y+z)^2 + x(y+z)]$
 $= x(y+z)(x+y+z)$

Putting these values in $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ we get,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{1}{x^4 z^4}$$

$$\begin{vmatrix} x+y+z & x+y+z & -(x+y) \\ -(x+y+z) & 0 & x \\ y(x-z)(x+y+z) & x(y+z)(x+y+z) & -xy(x+y) \end{vmatrix}$$

Taking $x+y+z$ common from C_1 as well as from C_2 we get,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{(x+y+z)^2}{x^4 z^4}$$

$$\begin{vmatrix} 1 & 1 & -(x+y) \\ -1 & 0 & x \\ y(x-z) & x(y+z) & -xy(x+y) \end{vmatrix}$$

$$= \frac{(x+y+z)^2}{x^4 z^4} \begin{vmatrix} 1 & 1 & -(x+y) \\ -1 & 0 & x \\ y(x-z) & x(y+z) & -xy(x+y) \end{vmatrix}$$

$$= \frac{(x+y+z)^2}{x^4 z^4} \begin{vmatrix} 0 & 1 & -y \\ -1 & 0 & x \\ y(x-z) & x(y+z) & -xy(x+y) \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2$

$$= \frac{(x+y+z)^2}{x^4 z^4}$$

$$[0 - 1\{xy(x+y) - xy(x-z)\} - y\{-x(y+z)\}]$$

$$= \frac{(x+y+z)^2}{x^4 z^4}$$

$$[-xy(x+y) + xy(x-z) + xy(y+z)]$$

$$= \frac{(x+y+z)^2}{x^4 z^4} \cdot xy [-x - y + x - z + y + z] = 0$$

$$\Rightarrow \frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$$

\Rightarrow Given function u, v and w are not independent

Find part: To find the relation,

we calculate uv

$$\text{Now, } uv = \frac{x+y}{z} \cdot \frac{y+z}{x} \left[\because u = \frac{x+y}{z}, v = \frac{y+z}{x} \right]$$

$$= \frac{(x+y)(y+z)}{xz} = \frac{xy + xz + y^2 + yz}{xz}$$

$$= \frac{y(x+y+z)}{xz} + \frac{xz}{xz}$$

$$= w + 1 \quad \left[\because w = \frac{y(x+y+z)}{xz} \right]$$

Thus, $uv = 2w + 1$, which is desired relation.

Q.2. (d) A rigid body is rotating with constant angular velocity ω about a fixed axis. If ' v ' is the linear velocity of any point of the body then prove that $\text{curl } v = 2\omega$.

Ans. Here, angular velocity of the body = $\vec{\omega}$

Linear velocity of the body = \vec{v}

$$\text{Let } \vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$$

We know that

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix} [\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}]$$

$$= \hat{i}(w_2z - w_3y) - \hat{j}(w_1z - w_3x) + \hat{k}(w_1y - w_2x)$$

Now,

$$\text{Curel } \vec{v} = v + \vec{v}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_2z - w_3y & w_3x - w_1z & w_1y - w_2x \end{vmatrix}$$

$$= \hat{i}[w_1 + w_1] - \hat{j}[-w_2 - w_2] + \hat{k}[w_3 + w_3]$$

$$= 2(w_1\hat{i} + w_2\hat{j} + w_3\hat{k})$$

$$\Rightarrow \text{Curel } \vec{v} = 2\vec{w} \quad [\because \vec{w} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}] \quad \text{Ans.}$$

Q 2. (e) Assuming $\sqrt{n|1-n|} = \pi \operatorname{cosec} n\pi$, $0 < n < 1$,

$$\text{show that } \int_0^\infty \frac{x^{p-1}}{1+x} dx = \left(\frac{\pi}{\sin p\pi} \right); 0 < p < 1$$

Ans. Given that $\sqrt{n|1-n|} = \pi \operatorname{cosec} n\pi$

To show

$$\text{We know that, } B(p, q) = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}} dx$$

$$\Rightarrow \frac{\sqrt{p|q|}}{\sqrt{p+q}} = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}} dx$$

$$[\because B(m, n) = \frac{\sqrt{m|n|}}{\sqrt{m+n}}]$$

$$\Rightarrow \frac{\sqrt{p|1-p|}}{\sqrt{p+1-p}} = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+1-p}} dx \quad [\text{Putting } q = 1-p]$$

$$\Rightarrow \frac{\sqrt{p|1-p|}}{1} = \int_0^\infty \frac{x^{p-1}}{1+x} dx$$

$$\Rightarrow \sqrt{p|1-p|} = \int_0^\infty \frac{x^{p-1}}{1+x} dx \quad [\because \pi = 1]$$

$$\Rightarrow \pi \operatorname{cosec} p\pi = \int_0^\infty \frac{x^{p-1}}{1+x} dx$$

$$[\because \sqrt{n|1-n|} = \pi \operatorname{cosec} n\pi]$$

$$\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi} \quad \text{Ans.}$$

Section-C

All questions of this section are compulsory. Attempt any two parts from each question :

(10 × 5) = 50

Q.3. (a) If $x = \sin\left(\frac{\log y}{a}\right)$ then evaluate the value

$$(1-x^2)y_{n+2} - (2n+1)x y_{n+1} - (n^2+a^2)y_n = 0$$

with usual symbols.

$$\text{Ans. We have, } x = \sin\left(\frac{\log y}{a}\right)$$

$$\Rightarrow \frac{\log y}{a} = a \sin^{-1} x$$

$$\Rightarrow y = e^{a \sin^{-1} x} \quad \dots(1)$$

$$\begin{aligned} [\because \log a &= x] \\ a &= e^x \end{aligned}$$

Differentiation (i) w. r. t x,

$$y_1 = e^{a \sin^{-1} x} \cdot \frac{d}{dx} (a \sin^{-1} x)$$

$$\Rightarrow y_1 = e^{a \sin^{-1} x} a \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2} \cdot y_1 = a \cdot e^{a \sin^{-1} x}$$

$$\Rightarrow \sqrt{1-x^2} y_1 = ay \quad [\because e^{a \sin^{-1} x} = y]$$

Squaring both sides, we get, $(1-x^2)y_1^2 = a^2 y^2$

$$\Rightarrow \frac{d}{dx} [(1-x^2)y_1^2] = \frac{d}{dx} [a^2 y^2]$$

$$\Rightarrow (1-x^2) \frac{d}{dx} (y_1^2) + (-2x) y_1^2 = a^2 \left(2y \frac{dy}{dx} \right)$$

$$(1-x^2) 2y_1 \frac{dy_1}{dx} - 2x y_1^2 = 2a^2 y y_1 \quad \left[\because \frac{dy}{dx} = y_1 \right]$$

$$\Rightarrow 2(1-x^2) y_1 y_2 - 2x y_1^2 = 2a^2 y y_1 \quad \left[\because \frac{dy_1}{dx} = y_2 \right]$$

$$\Rightarrow 2y_1 [(1-x^2) y_2 - x(y_1)] = 2y_1 [a^2 y]$$

$$\Rightarrow (1-x^2) y_2 - x y_1 - a^2 y = 0 \quad \dots (2)$$

Differentiating (2) n times by Labnitz's theorem, we get,

$$\left[(1-x^2) y_{n+2} + {}^n c_1 (-2x) (y_{n+1}) + {}^n c_2 (-2) y_n \right] - \left[x y_{n+1} + {}^n c_1 (1) y_n \right] - a^2 y_n = 0$$

$$\Rightarrow (1-x^2) y_{n+2} - 2nx y_{n+1} - \frac{n(n-1)}{2} \cdot 2y_n -$$

$$\left[x y_{n+1} + n y_n \right] - a^2 y_n = 0 \quad \left[\begin{array}{l} \because {}^n c_1 = n \text{ and} \\ {}^n c_2 = \frac{n(n-1)}{2} \end{array} \right]$$

$$\Rightarrow (1-x^2) y_{n+2} - 2nx y_{n+1} - \frac{n(n-1)}{2} y_n - x y_{n+1} - n y_n - a^2 y_n = 0$$

$$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} - [n(n-1) + n + a^2] y_n = 0$$

$$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2 + a^2) y_n = 0.$$

Hence Proved

Q.3. (b) If $u = \cos^{-1} \left(\frac{x+y}{\sqrt{x} + \sqrt{y}} \right)$, show that

$$x \frac{du}{dx} + y \frac{du}{dy} + \frac{1}{2} \cot u = 0$$

Ans. We have $u = \cos^{-1} \left(\frac{x+y}{\sqrt{x} + \sqrt{y}} \right)$

$$\Rightarrow \cos u = \frac{x+y}{\sqrt{x} + \sqrt{y}}$$

Let $v(x, y) = \frac{x+y}{\sqrt{x} + \sqrt{y}}$

$$= x_1 1 + \frac{\left(1 + \frac{y}{x} \right)}{\sqrt{x} \left(1 + \frac{\sqrt{y}}{\sqrt{x}} \right)}$$

$$= \frac{1}{x^{\frac{1}{2}}} f \left(\frac{y}{x} \right)$$

Thus, $v(x, y)$ is a homogeneous function of degree $+\frac{1}{2}$

\therefore By Euler's theorem, $x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = nV$

$$\Rightarrow x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = +\frac{1}{2} V \quad \left[\because n = +\frac{1}{2} \text{ here} \right]$$

$$\Rightarrow x \frac{\partial}{\partial x} (\cos u) + y \frac{\partial}{\partial y} (\cos u) = +\frac{1}{2} \cos u$$

$$\left[\because \frac{x+v}{\sqrt{x} + \sqrt{y}} = V = \cos u \right]$$

$$\Rightarrow x \frac{\partial}{\partial u} (\cos u) \cdot \frac{\partial u}{\partial x} + y \frac{\partial}{\partial u} (\cos u) \cdot \frac{\partial u}{\partial y} = +\frac{1}{2} \cos u$$

$$\Rightarrow x (-\sin u) \frac{\partial u}{\partial x} + y (-\sin u) \frac{\partial u}{\partial y} = +\frac{1}{2} \cos u$$

$$\Rightarrow -\sin u \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] = +\frac{1}{2} \cos u$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{2} \frac{\cos u}{\sin u}$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{2} \cot u \quad \left[\because \frac{\cos \theta}{\sin \theta} = \cot \theta \right]$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + \frac{1}{2} \cot u = 0 \quad \text{Hence Proved}$$

Q.3. (c) Verify Euler's theorem for

$$z = \frac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}$$

Ans. We have, $z = \frac{x^{\frac{1}{2}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} + y^{\frac{1}{2}}}$

To verify Euler's theorem for z , we shall first find degree of z .

$$\begin{aligned} \therefore z &= \frac{x^{\frac{1}{2}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} + y^{\frac{1}{2}}} \\ &= \frac{x^{\frac{1}{2}} \left(1 + \frac{y^{\frac{1}{3}}}{x^{\frac{1}{2}}} \right)}{x^{\frac{1}{2}} \left(1 + \frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}} \right)} \\ &= x^{\frac{1}{2} - \frac{1}{2}} \left[\frac{1 + \left(\frac{y}{x} \right)^{\frac{1}{3}}}{1 + \left(\frac{y}{x} \right)^{\frac{1}{2}}} \right] \\ &= x^{\frac{2-3}{6}} f\left(\frac{y}{x}\right) \left[\text{where } \frac{1 + \left(\frac{y}{x} \right)^{\frac{1}{3}}}{1 + \left(\frac{y}{x} \right)^{\frac{1}{2}}} = f\left(\frac{y}{x}\right) \right] \\ &= x^{-\frac{1}{6}} f\left(\frac{y}{x}\right) \end{aligned}$$

$\Rightarrow z$ is a homogeneous function of degree $-\frac{1}{6}$.

\therefore To verify Euler's theorem for z , we shall show,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = -\frac{1}{6} z$$

$$\left[\begin{array}{l} \because \text{Euler's theorem states} \\ x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz, \text{ when } n \text{ is degree} \\ \text{of homogeneous function } z \end{array} \right]$$

$$\begin{aligned} \text{Now, } \frac{\partial z}{\partial x} &= \frac{\partial}{\partial x} \left[\frac{x^{\frac{1}{2}} + y^{\frac{1}{3}}}{x^{\frac{1}{2}} + y^{\frac{1}{2}}} \right] \\ \frac{\partial z}{\partial x} &= \frac{\left(x^{\frac{1}{2}} + y^{\frac{1}{2}} \right) - \left(x^{\frac{1}{2}} + y^{\frac{1}{3}} \right) \left(\frac{1}{2} x^{-\frac{1}{2}} \right)}{\left(x^{\frac{1}{2}} + y^{\frac{1}{2}} \right)^2} \end{aligned}$$

$$x \frac{\partial z}{\partial x} = \frac{\frac{1}{3} \left(x^{\frac{1}{2}} + y^{\frac{1}{2}} \right) x^{\frac{1}{2}} - \frac{1}{2} \left(x^{\frac{1}{2}} + y^{\frac{1}{3}} \right) x^{\frac{1}{2}}}{\left(x^{\frac{1}{2}} + y^{\frac{1}{2}} \right)^2} \quad \dots(1)$$

Similarly,

$$y \frac{\partial z}{\partial y} = \frac{\frac{1}{3} \left(x^{\frac{1}{2}} + y^{\frac{1}{2}} \right) y^{\frac{1}{3}} - \frac{1}{2} \left(x^{\frac{1}{2}} + y^{\frac{1}{3}} \right) y^{\frac{1}{2}}}{\left(x^{\frac{1}{2}} + y^{\frac{1}{2}} \right)^2} \quad \dots(2)$$

(Replacing x by y in (1))

Adding (1) and (2),

$$\begin{aligned} &= x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \\ &= \frac{1}{\left(x^{\frac{1}{2}} + y^{\frac{1}{2}} \right)^2} \left[\frac{1}{3} \left(x^{\frac{1}{2}} + y^{\frac{1}{2}} \right) x^{\frac{1}{2}} - \frac{1}{2} \left(x^{\frac{1}{2}} + y^{\frac{1}{3}} \right) x^{\frac{1}{2}} \right. \\ &\quad \left. + \frac{1}{3} \left(x^{\frac{1}{2}} + y^{\frac{1}{2}} \right) y^{\frac{1}{3}} - \frac{1}{2} \left(x^{\frac{1}{2}} + y^{\frac{1}{3}} \right) y^{\frac{1}{2}} \right] \\ &= \frac{1}{\left(x^{\frac{1}{2}} + y^{\frac{1}{2}} \right)^2} \left[\frac{1}{3} \left(x^{\frac{1}{2}} + y^{\frac{1}{2}} \right) \left(x^{\frac{1}{2}} + y^{\frac{1}{3}} \right) - \frac{1}{2} \left(x^{\frac{1}{2}} + y^{\frac{1}{3}} \right) \left(x^{\frac{1}{2}} + y^{\frac{1}{2}} \right) \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{\left(\frac{1}{x^2} + \frac{1}{y^2}\right) \left(\frac{1}{x^3} + \frac{1}{y^3}\right) \left(\frac{1}{3} - \frac{1}{2}\right)}{\left(\frac{1}{x^2} + \frac{1}{y^2}\right)^2} \\
 &= \frac{\frac{1}{x^3} + \frac{1}{y^3}}{\frac{1}{x^2} + \frac{1}{y^2}} \left(\frac{2-3}{6}\right) \\
 &= -\frac{1}{6} \left(\frac{\frac{1}{x^3} + \frac{1}{y^3}}{\frac{1}{x^2} + \frac{1}{y^2}}\right) \\
 &= -\frac{1}{6} z \quad \left[\because z = \frac{\frac{1}{x^3} + \frac{1}{y^3}}{\frac{1}{x^2} + \frac{1}{y^2}} \right]
 \end{aligned}$$

Thus, $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = -\frac{1}{6} z$

Hence, Euler's theorem is verified.

Q.4. (a) In a plane ΔABC , find the maximum value of $\cos A \cos B \cos C$

Ans. In ΔABC , $A + B + C = \pi$

$$\Rightarrow C = \pi - (A + B)$$

Now,

$$\begin{aligned}
 \cos A \cos B \cos C &= \cos A \cos B \cos [\pi - (A + B)] \\
 &= -\cos A \cos B \cos (A + B) \quad [\because \cos(\pi - \theta) = -\cos \theta]
 \end{aligned}$$

$$\text{Let, } f = -\cos A \cos B \cos (A + B)$$

$$\begin{aligned}
 \Rightarrow \frac{\partial f}{\partial A} &= -\cos B \frac{\partial}{\partial A} [\cos A \cos (A + B)] \\
 &= -\cos B [-\sin A \cos (A + B) + \cos A \sin (A + B)] \\
 &= \cos B [-\sin (A + A + B)] \\
 [\because \sin A \cos B + \cos A \sin B &= \sin (A + B)]
 \end{aligned}$$

$$\Rightarrow \frac{\partial f}{\partial A} = \sin (2A + B) \cos B$$

$$\text{Similarly, } \frac{\partial f}{\partial B} = \sin (A + 2B) \cos A$$

[Interchanging A and B]

For maxima and minima

$$\frac{\partial f}{\partial A} = 0 \quad \frac{\partial f}{\partial B} = 0$$

$$\text{Now, } \frac{\partial f}{\partial A} = 0$$

$$\Rightarrow \sin (2A + B) \cos B = 0$$

$$\Rightarrow \sin (2A + B) = 0$$

$$\left[\because \cos B = 0 \Rightarrow B = \frac{\pi}{2}, \text{ which is not possible} \right]$$

$$\Rightarrow \sin (2A + B) = \sin \pi \quad [\because \sin \pi = 0]$$

$$\Rightarrow 2A + B = \pi \quad \dots(1)$$

Similarly,

$$\frac{\partial f}{\partial B} = 0 \text{ gives, } A + 2B = \pi \quad \dots(2)$$

$$\Rightarrow \text{Equation (1)} - 2 \times \text{equation (2) gives,}$$

$$B - 4B = \pi - 2\pi$$

$$\Rightarrow -3B = -\pi$$

$$\Rightarrow B = \frac{\pi}{3}$$

Putting value of B in (1), we get,

$$2A + \frac{\pi}{3} = \pi$$

$$\Rightarrow dA = \pi - \frac{\pi}{3}$$

$$\Rightarrow 2A = \frac{2\pi}{3}$$

$$\Rightarrow A = \frac{\pi}{3}$$

Thus, $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is the point where f can be maximum or minimum.

To check the nature of f at this point, we calculate, r_1 , s and t :

$$\text{Now, } r = \frac{\partial^2 f}{\partial A^2}$$

$$\Rightarrow r = 2 \cos (2A + B) \cos B$$

$$\text{Also, } s = \frac{\partial^2 f}{\partial A \partial B}$$

$$= \frac{\partial}{\partial A} \left[\frac{\partial f}{\partial B} \right] = \frac{\partial}{\partial A} [\sin(A + 2B) \cos A]$$

$$= \cos(A + 2B) \cos A - \sin(A + 2B) \sin A$$

$$= \cos(A + 2B + A)$$

$$[\because \cos(A + B) = \cos A \cos B - \sin A \sin B]$$

$$\Rightarrow s = \cos(2A + 2B)$$

Now,

$$t = \frac{\partial^2 f}{\partial B^2}$$

$$= \frac{\partial}{\partial B} [\sin(A + 2B) \cos A]$$

$$t = 2 \cos(A + 2B) \cos A$$

$$\text{At, } A = \frac{\pi}{3}, B = \frac{\pi}{3}$$

$$r = [2 \cos(2A + B) \cos B]_{\text{at } \left(\frac{\pi}{3}, \frac{\pi}{3}\right)}$$

$$= 2 \cos\left(\frac{2\pi}{3} + \pi\right) \cos \frac{\pi}{3}$$

$$= 2 \cos \pi \cos \frac{\pi}{3} = 2(-1) \cdot \frac{1}{2}$$

$$\Rightarrow \boxed{r = -1}$$

$$s = [\cos(2A + 2B) \cos B]_{\text{at } \left(\frac{\pi}{3}, \frac{\pi}{3}\right)}$$

$$= \cos\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right)$$

$$= \cos\left(\pi + \frac{\pi}{3}\right)$$

$$= -\cos \frac{\pi}{3} = -\frac{1}{2}$$

$$t = [2 \cos(A + 2B) \cos A]_{\text{at } \left(\frac{\pi}{3}, \frac{\pi}{3}\right)}$$

$$= 2 \cos\left(\pi + \frac{2\pi}{3}\right) \cos \frac{\pi}{3}$$

$$= 2 \cos \pi \cos \frac{\pi}{3}$$

$$= 2(-1) \left(\frac{1}{2}\right) = -1$$

$$\text{Now, } rt - s^2 = s^2 = (-1)(-1) - \left(-\frac{1}{2}\right)^2$$

$$= 1 - \frac{1}{4} = \frac{3}{4} \text{ (Positive)}$$

Also, $r = -1$ (Negative)

Thus, $rt - s^2$ is positive and r is negative

$\therefore f$ is maximum

Maximum value of $f =$

$$= -\cos \frac{\pi}{3} \cos \frac{\pi}{3} \cos\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \left(-\frac{1}{2}\right) = -\frac{1}{8}$$

Ans.

Q 4. (b) If $x = e^v \sec u$, $y = e^v \tan u$, then evaluate

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + \frac{1}{2} \cot u = 0$$

Ans. We have, $x = e^v \sec u$, $y = e^v \tan u$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} e^v \sec u \tan u & e^v \sec u \\ e^v \sec^2 u & e^v \tan u \end{vmatrix}$$

$$= e^v \sec u \cdot e^v \begin{vmatrix} \tan u & \sec u \\ \sec u & \tan u \end{vmatrix}$$

[taking $e^v \sec u$ common from c_1 and e^v from c_2]

$$= e^{2v} \sec u [\tan^2 u - \sec^2 u]$$

$$= -e^{2v} \sec u (1) \quad [\because \sec^2 \theta - \tan^2 \theta = 1]$$

$$\frac{\partial(x, y)}{\partial(u, v)} = -e^{2v} \sec u$$

Ans.

Q. 4. (c) The power 'P' required to propel a steamer of length 'P' at a speed 'u' is given by $P = \lambda u^3 P^3$ where λ is constant. If u is increased by 3% and P is decreased by 1%, find the corresponding increase in 'P'.

Ans. We are given,

$$P = \lambda u^3 P^3$$

...(i)

$$\therefore u \text{ is decreased by } 1\% \Rightarrow \frac{\partial l}{l} = -\frac{1}{100}$$

Taking log of both sides of (1), we get

$$\log \rho = \log \lambda + \log u^3 + \log l^3$$

$$[\because \log(m \cdot p) = \log m + \log n + \log p]$$

$$\Rightarrow \log \rho = \log \lambda + 3 \log u + 3 \log l$$

Taking differential, we get,

$$\Rightarrow \frac{\delta \rho}{\rho} = 0 + 3 \frac{\delta u}{u} + 3 \frac{\delta l}{l} \quad \left[\begin{array}{l} \because \lambda \text{ is constant} \\ \therefore \text{its differential is } 0 \end{array} \right]$$

$$\Rightarrow \frac{\delta \rho}{\rho} = \frac{9}{100} - \frac{3}{100}$$

$$\Rightarrow \frac{\delta \rho}{\rho} = -\frac{6}{100}$$

Thus ρ is increased by 6%.

Q.5. (a) Show that row vectors of the matrix

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \text{ are linearly independent.}$$

$$\text{Ans. Given matrix} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

To show that row vectors of given matrix are linearly independent, we take,

$$x_1 = (1, 2, -2), x_2 = (-1, 3, 0), x_3 = (0, -2, 1)$$

Consider the relation

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$$

$$\Rightarrow \lambda_1 (1, 2, -2) + \lambda_2 (-1, 3, 0) + \lambda_3 (0, -2, 1) = (0, 0, 0)$$

$$\Rightarrow 1\lambda_1 - 1\lambda_2 + 1\lambda_3 = 0$$

$$2\lambda_1 + 3\lambda_2 - 2\lambda_3 = 0$$

$$-2\lambda_1 - 0\lambda_2 + 1\lambda_3 = 0$$

$$\therefore \text{co-efficient matrix } A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 - 2R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_3 \rightarrow R_3 + 2R_2 \end{array}$$

Which is in echelon form.

\therefore Rank of A = Number of non zero row in echelon form = 3

\therefore Rank of $A = 3$ = Number of unknown

\Rightarrow System has only trivial soln.

$\Rightarrow l_1, l_2, l_3$ can have only zero value.

$\therefore X_1, X_2$ and X_3 are linearly independent. Thus, row vectors of given matrix are linearly independent.

Q.5. (b) Find the rank of the following matrix using elementary transformations

$$\begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$$

$$\text{Ans. Given matrix } A = \begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$$

To find the rank of A , we shall convert it into echelon form.

$$\text{Now, } A = \begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 13 & -2 & -8 \end{bmatrix} \begin{array}{l} R_3 \rightarrow R_3 - 3R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -28 & -47 \end{bmatrix} \begin{array}{l} R_3 \rightarrow R_3 - 13R_2 \end{array}$$

Which is in echelon form

∴ Rank of A = number of non zero rows in its echelon form. = 3

Ans.

Q.5. (c) Express the matrix

$$A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix} \text{ as a sum of Hermitian}$$

and Skew Hermitian matrix.

$$\text{Ans. Given matrix } A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$$

We know that for any square matrix A ,

$$A = \frac{1}{2}(A + A\theta) + \frac{1}{2}(A - A\theta) \quad \dots(1)$$

where, $A\theta = (\bar{A})$

Here, $(A - A\theta)$ is skew Hermitian matrix.

We shall calculate AQ now.

$$\therefore A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix} \quad \dots(2)$$

$$\Rightarrow \bar{A} = \begin{bmatrix} -i & 2+3i & 4-5i \\ 6-i & 0 & 4+5i \\ i & 2+i & 2-i \end{bmatrix} \text{ [Replacing } i \text{ by } -i]$$

$$\Rightarrow (\bar{A})' = \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4+5i & 4+5i & 2-i \end{bmatrix} \text{ [Interchanging rows and columns]}$$

$$\Rightarrow AQ = \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix} \quad (-3)$$

$$\text{Now, } A + AQ = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ i & 2-i & 2+i \end{bmatrix} + \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}$$

$$\Rightarrow A + AQ = \begin{bmatrix} 0 & 8-4i & 4+6i \\ 8+4i & 0 & 6-4i \\ 4-6i & 6+4i & 4 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2}(A + AQ) = \begin{bmatrix} 0 & 4-2i & 2+3i \\ 4+2i & 0 & 3-2i \\ 2-3i & 3+2i & 2 \end{bmatrix} \text{ [Dividing each element by 2]}$$

Again,

$$A - AQ = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ i & 2-i & 2+i \end{bmatrix}$$

$$- \begin{bmatrix} i & 6i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}$$

$$\Rightarrow (A - AQ) = - \begin{bmatrix} 2i & -4-2i & 4+4i \\ 4-2i & 0 & 2-6i \\ -4+4i & -2-6i & 2i \end{bmatrix}$$

$$\Rightarrow \frac{1}{2}(A - AQ) = \begin{bmatrix} i & -2-i & 2+2i \\ 2-i & 0 & 1-3i \\ -2+2i & -1-3i & i \end{bmatrix} \text{ [Dividing each element by 2]}$$

Putting, $\frac{1}{2}(A + AQ) =$ and $\frac{1}{2}(A - AQ)$ in (1), we get

$$A = \begin{bmatrix} 0 & 4-2i & 2+3i \\ 4+2i & 0 & 3-2i \\ 2-3i & 3+2i & 2 \end{bmatrix}$$

$$+ \begin{bmatrix} i & -2-i & 2+2i \\ 2-i & 0 & 1-3i \\ -2+2i & -1-3i & i \end{bmatrix}$$

Which is desired representation.

Q.6. (a) Interpret the physical meaning of $\text{curl } \vec{F}$ and $\text{div } \vec{F}$

Ans. Physical meaning of $\text{curl } \vec{F}$: Suppose a rigid body is rotating about a fixed axis through the origin

with linear velocity \vec{F} . Curl of \vec{F} gives 2 times the angular velocity of the body. Thus, curl is a measure of rotation.

Physical meaning of $\text{div } \vec{F}$: Consider the steady motion of a fluid having velocity \vec{F} . Rate of change of the volume of the fluid per unit time per unit volume is given by $\text{div } \vec{F}$.

Q.6. (b) Verify the divergence theorem for the function $\vec{F} = 4xz \hat{i} - y^2 \hat{j} + yz \hat{k}$; taken over the cube bounded by planes $x = 0, x = 1; y = 0, y = 1; z = 0, z = 1$.

Ans. Given function, $\vec{F} = 4xz \hat{i} - y^2 \hat{j} + yz \hat{k}$

To verify divergence theorem, we shall show that

$$\iiint_V \nabla \cdot \vec{F} \, dv = \iint_S \vec{F} \cdot \hat{n} \, ds, \text{ where } S \text{ is the given surface}$$

and V is the volume bounded by S .

Now,

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$[\text{If } \vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}]$$

$$\begin{aligned} \Rightarrow \nabla \cdot \vec{F} &= \frac{\partial}{\partial x} (4xz) + \frac{\partial}{\partial y} (-y^2) + \frac{\partial}{\partial z} (yz) \\ &= 4z - 2y + y \\ &= 4z - y \end{aligned}$$

$$\Rightarrow \iiint_V \nabla \cdot \vec{F} \, dv = \iiint_V (4z - y) \, dx \, dy \, dz$$

Now, V is the volume bounded by plane $x = 0, x = 1; y = 0, y = 1$ and $z = 0, z = 1$

\therefore Limits of x are 0 to 1

Limits of y are 0 to 1

Limits of z are 0 to 1

$$\begin{aligned} \Rightarrow \iiint_V \nabla \cdot \vec{F} \, dv &= \int_{x=0}^1 \int_{y=0}^1 \int_{z=0}^1 (4z - y) \, dz \, dy \, dx \\ &= \int_{x=0}^1 \int_{y=0}^1 \left[\frac{4z^2}{2} - yz \right]_{z=0}^1 \cdot dy \, dx \end{aligned}$$

$$= \int_{x=0}^1 \int_{y=0}^1 [2 - y] \, dy \, dx$$

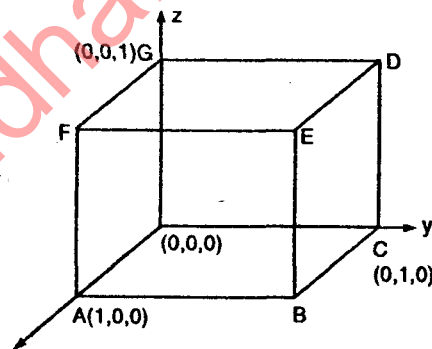
$$= \int_{x=0}^1 \left[2y - \frac{y^2}{2} \right]_{y=0}^1 \, dx$$

$$= \int_{x=0}^1 \left(2 - \frac{1}{2} \right) \, dx$$

$$= \frac{3}{2} \int_0^1 dx = \frac{3}{2}$$

$$\text{Thus, } \boxed{\iiint_V \nabla \cdot \vec{F} \, dv = \frac{3}{2}} \quad \dots(1)$$

The surface of given cube is shown in the figure



To calculate $\iiint_V \nabla \cdot \vec{F} \, dv$, we shall calculate surface

integral for each face of the cube and then their sum.

For this, we make the following table.

Surface	Equation	ds	\hat{n}	$\vec{F} \cdot \hat{n} = (4xz \hat{i} - y^2 \hat{j} + yz \hat{k}) \cdot \hat{n}$
OABC	$z = 0$	$dx \, dy$	$-\hat{k}$	$-yz = 0$
BCDE	$y = 1$	$dx \, dz$	\hat{j}	$-y^2 = 0$
DEFG	$z = 1$	$dx \, dy$	\hat{k}	$yz = y$
OCDG	$x = 0$	$dy \, dz$	$-\hat{i}$	$-4xz = 0$
AOGF	$y = 0$	$dx \, dz$	$-\hat{j}$	$y^2 = 0$
ABEF	$x = 1$	$dy \, dz$	\hat{i}	$4xz = 4z$

$$\text{Now, } \iint_{OABC} \vec{F} \cdot \hat{n} \, ds = \iint_{OABC} 0 \, dx \, dy = 0$$

$$\iint_{BCDE} \vec{F} \cdot \hat{n} \, ds = \iint_{BCDE} (-1) \, dx \, dz$$

$$= - \int_{x=0}^1 \int_{z=0}^1 dx \, dz$$

$$= - \int_{x=0}^1 [z]_0^1 dx$$

$$= - \int_{x=0}^1 dx = -1$$

$$\iint_{DEFG} \vec{F} \cdot \hat{n} \, ds = \iint_{DEFG} y \, dx \, dy$$

$$= \int_{x=0}^1 \int_{y=0}^1 y \, dy \, dx$$

$$= \int_{x=0}^1 \left[\frac{y^2}{2} \right]_{y=0}^1 dx$$

$$= \int_{x=0}^1 \frac{1}{2} dx$$

$$= \frac{1}{2} [x]_0^1 = \frac{1}{2}$$

$$= \iint_{OC DG} \vec{F} \cdot \hat{n} \, ds = \iint_{OC DG} 0 \, dy \, dz = 0$$

$$= \iint_{AOGF} \vec{F} \cdot \hat{n} \, ds = \iint_{AOGF} 0 \, dx \, dz = 0$$

$$= \iint_{ABEF} \vec{F} \cdot \hat{n} \, ds = \iint_{ABEF} 4z \, dy \, dz$$

$$= \int_{y=0}^1 \int_{z=0}^1 4z \, dz \, dy$$

$$= 4 \int_{y=0}^1 \left[\frac{z^2}{2} \right]_0^1 dy$$

$$= \frac{4}{2} \int_{y=0}^1 dy = 2$$

Adding all above integrals,

$$\iint_{OABC} \vec{F} \cdot \hat{n} \, ds + \iint_{BCDE} \vec{F} \cdot \hat{n} \, ds + \iint_{DEFG} \vec{F} \cdot \hat{n} \, ds$$

$$+ \iint_{OC DG} \vec{F} \cdot \hat{n} \, ds + \iint_{AOGF} \vec{F} \cdot \hat{n} \, ds + \iint_{ABEF} \vec{F} \cdot \hat{n} \, ds$$

$$= 0 - 1 + \frac{1}{2} + 0 + 0 + 2$$

$$\Rightarrow \iint_S \vec{F} \cdot \hat{n} \, ds = \frac{3}{2}$$

$$\text{Thus, } \boxed{\iiint_V \nabla \cdot \vec{F} \, dv = \iint_S \vec{F} \cdot \hat{n} \, ds}$$

Hence, divergence theorem is verified.

Q.6. (c) If a vector field is given by

$\vec{F} = (x^2 - y^2 + x) \hat{i} - (2xy + y) \hat{j}$. Is this field irrotational? If so, find its scalar potential.

Ans. Given vector field

$$\vec{F} = (x^2 - y^2 + x) \hat{i} - (2xy + y) \hat{j}$$

To check that \vec{F} is irrotational, we shall find $\nabla \times \vec{F}$.

Now,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\Rightarrow \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 + x & -2xy - y & 0 \end{vmatrix}$$

$$= \hat{i} [0 - 0] - \hat{j} [0 - 0] + \hat{k} [-2y + 2y]$$

$$\text{Thus, } \nabla \times \vec{F} = 0$$

$\Rightarrow \vec{F}$ is irrotational. Let ϕ be the scalar potential

$$\Rightarrow \vec{F} = \nabla \phi$$

$$\Rightarrow \vec{F} \cdot d\vec{r}$$

$$\Rightarrow d\phi = \vec{F} \cdot d\vec{r} \quad [\because \nabla \phi \cdot d\vec{r} = d\phi]$$

$$\Rightarrow d\phi = [(x^2 - y^2 + x) \hat{i} - (2xy + y) \hat{j}] \cdot d\vec{r}$$

$$\Rightarrow d\phi = (x^2 - y^2 + x) dx - (2xy + y) dy$$

$$\begin{aligned} \vec{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ \Rightarrow d\vec{r} &= dx\hat{i} + dy\hat{j} + dz\hat{k} \\ \Rightarrow d\phi &= x^2 dx - y^2 dx + xdx - 2x y dy - ydy \\ \Rightarrow d\phi &= x^2 dx + xdx - ydy - (y^2 s + 2x y dy) \\ \Rightarrow d\phi &= d\left(\frac{x^3}{3}\right) + d\left(\frac{x^2}{2}\right) - d\left(\frac{y^2}{2}\right) - d(xy^2) \end{aligned}$$

Integrating

$$\phi = \frac{x^3}{3} + \frac{x^2}{2} - \frac{y^2}{2} - xy^2 + c$$

Ans.

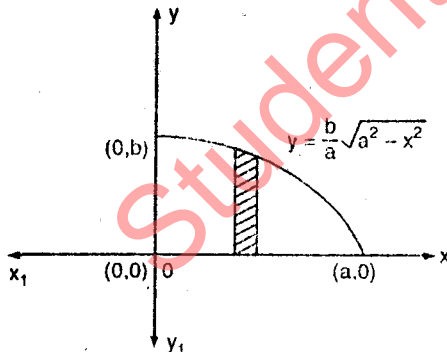
Q.7. (a) Evaluate $\iint_R \left(1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}\right) dx dy$ over

the first quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Ans. Given integral $I = \iint_R \left(1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}\right) dx dy$

Region of integration is positive quadrant of the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which is shown in figure.



Limits of x are from 0 to a

Limits of y are from 0 to $\frac{b}{a} \sqrt{a^2 - x^2}$

\Rightarrow Given integral,

$$I = \int_{x=0}^a \int_{y=0}^{\frac{b}{a} \sqrt{a^2 - x^2}} \left(1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}\right) dx dy$$

$$\begin{aligned} &= \int_{x=0}^a \left[\left(1 - \frac{x^2}{a^2}\right) y + \frac{y^3}{3b^2} \right]_{y=0}^{\frac{b}{a} \sqrt{a^2 - x^2}} dx \\ &= \int_{x=0}^a \left[\left(1 - \frac{x^2}{a^2}\right) \frac{b}{a} \sqrt{a^2 - x^2} + \frac{1}{3b^2} \left(\frac{b}{a} \sqrt{a^2 - x^2}\right)^3 \right] dx \end{aligned}$$

$$\begin{aligned} &= \int_{x=0}^a \left[\frac{1}{a^2} (a^2 - x^2) \frac{b}{a} \sqrt{a^2 - x^2} + \frac{b^3}{3a^3 b^2} \left(a^2 - x^2, \frac{3}{2}\right) \right] dx \end{aligned}$$

$$\begin{aligned} &= \int_{x=0}^a \left[\frac{b}{a^3} (a^2 - x^2)^{\frac{3}{2}} + \frac{b}{3a^3} \left(a^2 - x^2, \frac{3}{2}\right) \right] dx \end{aligned}$$

$$\begin{aligned} &= \int_{x=0}^a \frac{b}{3a^3} (a^2 - x^2)^{\frac{3}{2}} (3+1) dx \end{aligned}$$

$$\Rightarrow I = \frac{4b}{3a^3} \int_0^a (a^2 - x^2)^{3/2} dx$$

Put, $x = a \sin \theta$

$\Rightarrow dx = a \cos \theta d\theta$

at $x=0$, $\sin \theta = 0 \Rightarrow \theta = 0$

at $x=a$, $\sin \theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\Rightarrow I = \frac{4b}{3a^3} \int_0^{\frac{\pi}{2}} (a^2 - a^2 \sin^2 \theta)^{3/2} a \cos \theta d\theta$$

$$\begin{aligned} &= \frac{4b}{3a^2} \int_0^{\frac{\pi}{2}} [a^2 (1 - \sin^2 \theta)]^{3/2} \cos \theta d\theta \end{aligned}$$

$$\begin{aligned} &= \frac{4b}{3a^2} \int_0^{\frac{\pi}{2}} a^3 \cos^3 \theta \cos \theta d\theta \end{aligned}$$

$$\begin{aligned} &= \frac{4}{3} ab \int_0^{\frac{\pi}{2}} \sin \theta \cos 4\theta d\theta \end{aligned}$$

$$= \frac{4}{3} ab \frac{\frac{0+1}{2} \frac{4+1}{2}}{\frac{0+4+2}{2}} \left[\frac{\frac{p+1}{2} \frac{q+1}{2}}{\frac{p+q+2}{2}} \right] \int_0^{\frac{\pi}{2}} \sin \theta \cos^2 \theta d\theta$$

$$= \frac{2}{3} ab \frac{\frac{1}{2} \frac{5}{2}}{\frac{3}{2}} \left[\frac{1}{2} = \sqrt{\pi} \right]$$

$$= \frac{2}{3} ab \frac{\sqrt{\pi} \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}}{2!} \left[\frac{1}{2} = \sqrt{\pi} \right]$$

$$= \frac{\pi ab}{4}$$

Ans.

Q.7. (b) Find the mass of the region bounded by ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; if the density varies as the square of the distance from the centre.

Ans. Given ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Let ρ be the density at a point (x, y, z)

\therefore Density varies as the square of the distance from centre.

\therefore We shall find distance of (x, y, z) from centre $(0, 0, 0)$. By distance formula, distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

$$\Rightarrow d = \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2}$$

$$\Rightarrow d^2 = x^2 + y^2 + z^2$$

$$\therefore f = kd^2 \text{ (where } k \text{ is constant)}$$

$$\Rightarrow \delta = k(x^2 + y^2 + z^2)$$

Now, mass = $\iiint_V \rho \, dx \, dy \, dz$, where r is the volume

of given ellipsoidal.

$$= k \iiint_V (x^2 + y^2 + z^2) \, dx \, dy \, dz$$

Putting, $x = ar \sin \theta \cos \phi$

$y = br \sin \theta \sin \phi$

$z = cr \cos \theta$

[changing to ellipsoidal polar coordinates]

$$dx \, dy \, dz = abc \, r^2 \sin \theta \, dr \, d\theta \, d\phi$$

Limits of r are 0 to 1

Limits of θ are 0 to π

Limits of ϕ are 0 to 2π

\Rightarrow Mass

$$= k \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{r=0}^1 [a^2 r^2 \sin^2 \theta \cos^2 \phi + b^2 r^2 \sin^2 \theta \sin^2 \phi + c^2 r^2 \cos^2 \theta] abc \, r^2 \sin \theta \, dr \, d\theta \, d\phi$$

$$= kabc \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{r=0}^1 [a^2 \sin^2 \theta \cos^2 \phi + b^2 \sin^2 \theta \sin^2 \phi + c^2 \cos^2 \theta] \sin \theta \, r^4 \, dr \, d\theta \, d\phi$$

$$= kabc \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} [(a^2 \sin^2 \theta \cos^2 \phi + b^2 \sin^2 \theta \sin^2 \phi + c^2 \cos^2 \theta) \sin \theta \frac{r^5}{5}]_{r=0}^1 d\theta \, d\phi$$

$$= \frac{kabc}{5} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} [(a^2 \sin^2 \theta \cos^2 \phi + b^2 \sin^2 \theta \sin^2 \phi + c^2 \cos^2 \theta) \sin \theta] d\theta \, d\phi$$

$$= \frac{kabc}{5} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left[a^2 \sin^2 \theta \left(\frac{1 + \cos 2\phi}{2} \right) + b^2 \sin^2 \theta \left(\frac{1 - \cos 2\phi}{2} \right) + c^2 \cos^2 \theta \right] \sin \theta \, d\phi \, d\theta$$

$$= \frac{kabc}{5} \int_{\theta=0}^{\pi} \left[\frac{a^2 \sin^2 \theta}{2} \left(\theta + \frac{\sin 2\phi}{2} \right) + \frac{b^2 \sin^2 \theta}{2} \left(\theta - \frac{\sin 2\phi}{2} \right) + c^2 \cos^2 \theta \cdot \phi \right]_{\phi=0}^{2\pi} \sin \theta \, d\theta$$

$$\left(\phi - \frac{\sin 2\phi}{2} \right) + c^2 \cos^2 \theta \cdot \phi \Big|_{\phi=0}^{2\pi} \sin \theta \, d\theta$$

$$= \frac{kabc}{5} \int_{\theta=0}^{\pi} \left[\frac{a^2 \sin^2 \theta}{2} (2\pi) + \frac{b^2 \sin^2 \theta}{2} (2\pi) \right. \\ \left. (2\pi) + c^2 \cos^2 \theta (2\pi) \right] \sin \theta d\theta \\ = \pi \frac{kabc}{5} \int_{\theta=0}^{\pi} [(a^2 + b^2) \sin^2 \theta + 2c^2 \cos^2 \theta] \sin \theta d\theta$$

$$= \pi \frac{kabc}{5} \int_{\theta=0}^{\pi} [(x^2 + b^2) - (a^2 + b^2) \cos^2 \theta + 2c^2 \cos^2 \theta] \sin \theta d\theta$$

$$= \pi \frac{kabc}{5} \int_{\theta=0}^{\pi} [(a^2 + b^2) \sin \theta d\theta + \pi \frac{kabc}{5} \int_{\theta=0}^{\pi} (2c^2 - a^2 - b^2) \cos^2 \theta \sin \theta d\theta$$

$$= \pi \frac{kabc}{5} (a^2 + b^2) (-\cos \theta)_0^{\pi} +$$

$$= \pi \frac{kabc}{5} (2c^2 - a^2 - b^2) I,$$

$$\text{where } I = \int_0^{\pi} \cos^2 \theta \sin \theta d\theta$$

$$= \frac{\pi kabc}{5} (a^2 + b^2) (2) + \frac{\pi kabc}{5} (2c^2 - a^2 - b^2) I$$

Now, we shall calculate I

$$\therefore I = \int_0^{\pi} \cos^2 \theta \sin \theta d\theta$$

$$= - \int_1^{-1} t^2 dt$$

$$= - \left[\frac{t^3}{3} \right]_1^{-1}$$

$$= - \left[\frac{-1}{3} - \frac{1}{3} \right] = \frac{2}{3}$$

$$\Rightarrow \text{Mass} = 2\pi \frac{kabc}{5} (a^2 + b^2) +$$

$$\frac{\pi kabc}{5} (2c^2 - a^2 - b^2) \frac{2}{3}$$

$$= 2\pi \frac{kabc}{5} \left[a^2 + b^2 + \frac{2c^2 - a^2 - b^2}{3} \right]$$

$$= 2\pi \frac{kabc}{15} [3a^2 + 3b^2 + 2c^2 - a^2 - b^2]$$

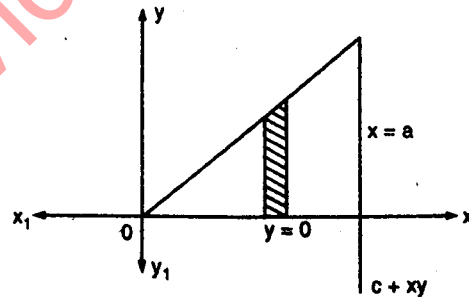
$$= \frac{4\pi kabc}{15} (a^2 + b^2 + c^2)$$

$$\therefore \text{Required mass} = \frac{4\pi kabc}{15} (a^2 + b^2 + c^2) \text{ Ans.}$$

Q.7. (c) A triangular prism is formed by the planes whose equations are $ay = bx$, $y = 0$ and $x = a$. Find the volume of this prism between the plane $z = 0$ and the surface $z = c + xy$.

$$\text{Ans. Required volume} = \iiint_V dx dy dz$$

where, V is the given region



Clearly limits of z are 0 to $c + xy$

Limits of y are 0 to $\frac{bx}{a}$

Limits of x are 0 to a

$$\therefore \text{Required volume} = \int_{x=0}^a \int_{y=0}^{\frac{bx}{a}} \int_{z=0}^{c+xy} dz dy dx$$

$$= \int_{x=0}^a \int_{y=0}^{\frac{bx}{a}} [z]_{z=0}^{c+xy} dy dx$$

$$= \int_{x=0}^a \int_{y=0}^{\frac{bx}{a}} (c + xy) dy dx$$

$$= \int_{x=0}^a \left[cy + \frac{xy^2}{2} \right]_{y=0}^{\frac{bx}{a}} dx$$

$$= \int_{x=0}^a \left[\frac{cbx}{a} + \frac{x^2}{2} \left(\frac{bx}{a} \right)^2 \right] dx$$

$$= \int_{x=0}^a \left(\frac{cbx}{a} + \frac{b^2 x^3}{2 a^2} \right) dx$$

$$= \left[\frac{cb}{a} \frac{x^2}{2} + \frac{b^2}{2 a^2} \frac{x^4}{4} \right]_{x=0}^a$$

$$= \frac{cb}{a} \left(\frac{a^2}{2} \right) + \frac{b^2}{8 a^2} a^4$$

$$= \frac{abc}{2} + \frac{a^2 b^2}{8} \quad \text{Ans.}$$